MOTION OF A CENTER OF MASS

Consider a system of particles where the position of the CM is given by:

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{M}$$

Taking the derivative wrt time gives:

$$\frac{d\vec{r}_{cm}}{dt} = \frac{d}{dt}\left[\frac{\sum m_i \vec{r}_i}{M}\right]$$

$$\vec{v}_{cm} = \frac{1}{M}\sum m_i \frac{d\vec{r}_i}{dt} = \frac{1}{M}\sum m_i \vec{v}_i$$

$$\vec{v}_i = velocity\ of\ \ i^{th}\ \ particle$$

$$M\vec{v}_{cm} = \sum m_i \vec{v}_i = \sum \vec{p}_i = \vec{P}_{sys}$$

$\boxed{\vec{P}_{sys} = M\vec{v}_{cm}}$ Total Linear Momentum of a System

1. The total linear momentum of the system is equal to the total mass of the system times the velocity of the center of mass of the system.

2. The total linear momentum of the system is equal to that of a single particle of mass M located at the center of mass and moving with a velocity equal to that of the center of mass.

$$\frac{d\vec{v}_{cm}}{dt} = \frac{d}{dt}\left[\frac{1}{M}\sum m_i \vec{v}_i\right]$$

$$\vec{a}_{cm} = \frac{1}{M}\sum m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum m_i \vec{a}_i$$

$$M\vec{a}_{cm} = \sum m_i \vec{a}_i = \sum \vec{F}_i$$

$$\vec{F}_i = \vec{F}_{ext} + \vec{F}_{\text{int}}$$

$\vec{F}_i$ = net force on i^{th} particle

$\vec{F}_{ext}$ = net external force on i^{th} particle

$\vec{F}_{\text{int}}$ = net internal force on i^{th} particle

$$M\vec{a}_{cm} = \sum \vec{F}_{ext} + \sum \vec{F}_{\text{int}}$$

$$By\ N3L\ \sum \vec{F}_{\text{int}} = 0$$

Therefore,

$$\sum \vec{F}_{ext} = M\vec{a}_{cm}$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = \frac{d}{dt}\ M\vec{v}_{cm} = \frac{d\vec{P}_{sys}}{dt}$$

$$\boxed{\sum \vec{F}_{ext} = M\vec{a}_{cm} = \frac{d\vec{P}_{sys}}{dt}}$$ N2L for a System

A system moves as if all the mass was concentrated on a single particle of mass M located at the center of mass and the net external force were applied to that point.

If $\sum \vec{\mathrm{F}}_{\text{ext}} = 0$, then :

$$\boxed{\begin{aligned}\vec{P}_{sys} &= M\vec{V}_{cm} = \text{constant} \\ \vec{P}_i &= \vec{P}_f\end{aligned}}$$ Law of Conservation of Linear Momentum for a System

Thus,

If $\sum \vec{F}_{ext} = 0,$ then

$\vec{P}_{sys}$ and $\vec{V}_{cm}$ are constants in time.