a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m = \frac{34 - 24.5}{30 - 0} = \frac{9.5}{30} = 0.32$$

The y intercept occurs when $x = 0$, so b = 24.5

$$y = 0.32x + 24.5$$

b. Now to predict the population in the year 2025, we let $x = 2025 - 1980 = 45$

$$y = 0.32x + 24.5$$

$$y = 0.32(45) + 24.5 = 38.9$$

In the year 2025, we predict that the population of Canada will be 38.9 million people.

Note that we assumed the population trend will continue to be linear. Therefore if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

### SECTION 3.2 PROBLEM SET: MINIMIZATION APPLICATIONS

For each of the following minimization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. Food A costs $2 per unit and Food B cost $1 per unit. How many units of each food should be purchased to keep costs at a minimum?

2) A computer store sells two types of computers, laptops and desktops. The supplier demands that at least 150 computers be sold a month. Experience shows that most consumers prefer laptops, but some business customers require desktops. The result is that the number of laptops sold is at least twice of the number of desktops. The store pays its sales staff a $60 commission for each laptop, and a $40 commission for each desktop. Let x = the number of laptops and y = the number of desktop computers. How many of each type must be sold to minimize commission to its sales people? What is the minimum commission?

We show the results as follows:

| Frequency of compounding | Formula | Total amount |
|---|---|---|
| Annually | $1(1 + 1) | $2 |
| Semiannually | $\$1(1 + 1/2)^{2}$ | $2.25 |
| Quarterly | $\$1(1 + 1/4)^{4}$ | $2.44140625 |
| Monthly | $\$1(1 + 1/12)^{12}$ | $2.61303529 |
| Daily | $\$1(1 + 1/365)^{365}$ | $2.71456748 |
| Hourly | $\$1(1 + 1/8760)^{8760}$ | $2.71812699 |
| Every minute | $\$1(1 + 1/525600)^{525600}$ | $2.71827922 |
| Every Second | $\$1(1 + 1/31536000)^{31536000}$ | $2.71828247 |
| Continuously | $1(2.718281828...) | $2.718281828... |

We have noticed that the $1 we invested does not grow without bound. It starts to stabilize to an irrational number 2.718281828... given the name "*e*" after the great mathematician Euler.

In mathematics, we say that as n becomes infinitely large the expression $\left(1+\frac{1}{n}\right)^{n}$ equals *e*.

Therefore, it is natural that the number e play a part in continuous compounding.

It can be shown that as n becomes infinitely large the expression $\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$

Therefore, it follows that if we invest $P at an interest rate r per year, compounded continuously, after t years the final amount will be given by

$$A = P \cdot e^{rt}.$$

◆***Example 6*** $3500 is invested at 9% compounded continuously. Find the future value in 4years.

***Solution:*** Using the formula for the continuous compounding, we get $A = Pe^{rt}$.

$A = \$3500e^{.09\times4}$

$A = \$3500e^{.36}$

$A = \$5016.65$

◆***Example 7*** If an amount is invested at 7% compounded continuously, what is the effective interest rate?

***Solution:*** If we deposit $1 in the bank at 7% compounded continuously for one year, and subtract that $1 from the final amount, we get the effective interest rate in decimals.

$r_{EFF} = 1e^{.07} - 1$

$r_{EFF} = 1.0725 - 1$

$r_{EFF} = .0725$ or 7.25%

The fourth payment will accumulate to $\$500(1 + .08/12)^{56}$.

And so on . . .

Finally the next to last (59th) payment will accumulate to $\$500(1 + .08/12)^{1}$.

The last payment is taken out the same time it is made, and will not earn any interest.

To find the total amount in five years, we need to add the accumulated value of these sixty payments.

In other words, we need to find the sum of the following series.

$$\$500(1 + .08/12)^{59} + \$500(1 + .08/12)^{58} + \$500(1 + .08/12)^{57} + \ldots + \$500$$

Written backwards, we have

$$\$500 + \$500(1 + .08/12) + \$500(1 + .08/12)^{2} + \ldots + \$500(1 + .08/12)^{59}$$

This is a geometric series with a = \$500, r = (1 + .08/12), and n = 59. The sum is

$$\frac{\$500[(1 + .08/12)^{60} - 1]}{.08/12}$$

$$= \$500(73.47686)$$

$$= \$36738.43$$

When the payments are made at the end of each period rather than at the beginning, we call it an **ordinary annuity.**

**Future Value of an Ordinary Annuity**

If a payment of m dollars is made in an account n times a year at an interest r, then the final amount A after t years is

$$\mathbf{A = \frac{m[(1 + r/n)^{nt} - 1]}{r/n}}$$

The future value is also called the accumulated value

◆***Example 2*** Tanya deposits \$300 at the end of each quarter in her savings account. If the account earns 5.75% compounded quarterly, how much money will she have in 4 years?

***Solution:*** The future value of this annuity can be found using the above formula.

$$A = \frac{\$300[(1 + .0575/4)^{16} - 1]}{.0575/4}$$

$$A = \$300(17.8463) = \$5353.89$$

If Tanya deposits \$300 into a savings account earning 5.75% compounded quarterly for 4 years, then at the end of 4 years she will have \$5,353.89

◆***Example 3*** Robert needs $5,000 in three years. How much should he deposit each month in an account that pays 8% compounded monthly in order to achieve his goal?

***Solution:*** If Robert saves m dollars per month, after three years he will have

$$\frac{m[(1+.08/12)^{36}-1]}{.08/12}$$

But we'd like this amount to be $5,000. Therefore,

$$\frac{m[(1+.08/12)^{36}-1]}{.08/12} = \$5000$$

$$m\,(40.5356) = \$5000$$

$$m = \frac{5000}{40.5356} = \$123.35$$

Robert needs to deposit $123.35 at the end of each month for 3 years into an account paying 8% compounded monthly in order to have $5,000 at the end of 5 years.

***SINKING FUND***

When a business deposits money at regular intervals into an account in order to save for a future purchase of equipment, the savings fund is referred to as a "**sinking fund**". Calculating the sinking fund deposit uses the same method as the previous problem.

◆***Example 4*** A business needs $450,000 in five years. How much should be deposited each quarter in a sinking fund that earns 9% compounded quarterly to have this amount in five years?

***Solution:*** Again, suppose that m dollars are deposited each quarter in the sinking fund. After five years, the future value of the fund should be $450,000. This suggests the following relationship:

$$\frac{m\,[(1+.09/4)^{20}-1]}{.09/4} = \$450{,}000$$

$$m\,(24.9115) = 450{,}000$$

$$m = \frac{450000}{24.9115} = \$\,18{,}063.93$$

The business needs to deposit $18063.93 at the end of each quarter for 5 years into an sinking fund earning interest of 9% compounded quarterly in order to have $450,000 at the end of 5 years.

***PROBLEMS INVOLVING MULTIPLE STAGES OF SAVINGS AND/OR ANNUITIES***

Consider the following situations:

a. Suppose a baby, Aisha, is born and her grandparents invest $8000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to need to finish college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?

b. Aisha graduates college and starts a job. She saves $1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

These problems appear complicated. But each can be broken down into two smaller problems involving compound interest on savings or involving annuities. Often the problem involves a savings period followed by an annuity period. ; the accumulated value from first part of the problem may become a present value in the second part. Read each problem carefully to determine what is needed.

◆***Example 2*** Suppose a baby, Aisha, is born and her grandparents invest $8000 in a college fund. The money remains invested for 18 years until Aisha enters college; then it is withdrawn in equal semiannual payments over the 4 years that Aisha expects to attend college. The college investment fund earns 5% interest compounded semiannually. How much can Aisha withdraw from the account every six months while she is in college?

***Solution:*** **Part 1: Accumulation of College Savings:** Find the accumulated value at the end of 18 years of a sum of $8000 invested at 5% compounded semiannually.

$$A = \$8000(1 + .05/2)^{(2\times 18)} = \$8000(1.025)^{36} = \$8000(2.432535)$$

$$A = \$19460.28$$

**Part 2: Seminannual annuity payout from savings to put toward college expenses.** Find the amount of the semiannual payout for four years using the accumulated savings from part 1 of the problem with an interest rate of 5% compounded semiannually.

A= $19460.28 in Part 1 is the accumulated value at the end of the savings period. This ecomes the present value P =$19460.28 when calculating the semiannual payments in Part 2.

$$\$19460.28\left(1+\frac{.05}{2}\right)^{2\times 4} = \frac{m\left[\left(1+\frac{.05}{2}\right)^{2\times 4} - 1\right]}{(.05/2)}$$

$$\$23710.46 = m\,(8.73612)$$

$$m = \$2714.07$$

Aisha will be able to withdraw $2714.07 semiannually for her college expenses.

◆***Example 5*** The transition matrix for Example 3 is given below.

| | | Tuesday | |
|---|---|---|---|
| | | Walk | Bicycle |
| **Monday** | Walk | 1/2 | 1/2 |
| | Bicycle | 1/4 | 3/4 |

Write the transition matrix from a) Monday to Thursday, b) Monday to Friday.

***Solution:*** In writing a transition matrix from Monday to Thursday, we are moving from one state to another in three steps. That is, we need to compute $T^3$.

$$T^3 = \begin{bmatrix} 11/32 & 21/32 \\ 21/64 & 43/64 \end{bmatrix}$$

b) To find the transition matrix from Monday to Friday, we are moving from one state to another in 4 steps. Therefore, we compute $T^4$.

$$T^4 = \begin{bmatrix} 43/128 & 85/128 \\ 85/256 & 171/256 \end{bmatrix}$$

It is important that the student is able to interpret the above matrix correctly. For example, the entry 85/128, states that if Professor Symons walked to school on Monday, then there is 85/128 probability that he will bicycle to school on Friday.

There are certain Markov chains that tend to stabilize in the long run. We will examine these more deeply later in this chapter. The transition matrix we have used in the above example is just such a Markov chain. The next example deals with the long term trend or steady-state situation for that matrix.

◆***Example 6*** Suppose Professor Symons continues to walk and bicycle according to the transition matrix given in Example 3. In the long run, how often will he walk to school, and how often will he bicycle?

***Solution:*** If we examine higher powers of the transition matrix T, we will find that it stabilizes.

$$T^5 = \begin{bmatrix} .333984 & 666015 \\ .333007 & .666992 \end{bmatrix} \qquad T^{10} = \begin{bmatrix} .33333397 & .66666603 \\ .33333301 & .66666698 \end{bmatrix}$$

And $$T^{20} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix}$$ and $$T^n = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} \text{ for } n > 20$$

The matrix shows that in the long run, Professor Symons will walk to school 1/3 of the time and bicycle 2/3 of the time.

When this happens, we say that the system is in steady-state or state of equilibrium. In this situation, all row vectors are equal. If the original matrix is an n by n matrix, we get n row vectors that are all the same. We call this vector a **fixed probability vector** or the **equilibrium vector** E. In the above problem, the fixed probability vector E is [1/3 2/3]. Furthermore, if the equilibrium vector E is multiplied by the original matrix T, the result is the equilibrium vector E. That is,

$$ET = E \text{ , or } \quad \begin{bmatrix} 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

◆***Question 2*** Does the product of an equilibrium vector and its transition matrix always equal the equilibrium vector? That is, does ET = E?

***Answer:*** At this point, the reader may have already guessed that the answer is yes if the transition matrix is a regular Markov chain. We try to illustrate with the following example from section 10.1.

A city is served by two cable TV companies, BestTV and CableCast. Due to their aggressive sales tactics, each year 40% of BestTV customers switch to CableCast; the other 60% of BestTV customers stay with BestTV. On the other hand, 30% of the CableCast customers switch to Best RV and 70% of CableCast customers stay with CableCast.

The transition matrix is given below.

| | | **Next Year** | |
|---|---|---|---|
| | | BestTV | CableCast |
| **Initial Year** | BestTV | .60 | .40 |
| | CableCast | .30 | .70 |

If the initial market share for BestTV is 20% and for CableCast is 80%, we'd like to know the long term market share for each company.

Let matrix T denote the transition matrix for this Markov chain, and $V_0$ denote the matrix that represents the initial market share. Then $V_0$ and T are as follows:

$$V_0 = [\ .20 \quad .80\ ] \quad \text{and} \quad T = \begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix}$$

Since each year people switch according to the transition matrix T, after one year the distribution for each company is as follows:

$$V_1 = V_0\, T = [\ .20 \quad .80\ ] \begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = [\ .36 \quad .64\ ]$$

After two years, the market share for each company is

$$V_2 = V_1\, T = [\ .36 \quad .64\ ] \begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = [\ .408 \quad .592\ ]$$

After three years the distribution is

$$V_3 = V_2\, T = [\ .408 \quad .592\ ] \begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = [\ .4224 \quad .5776\ ]$$

After 20 years the market share are given by $V_{20} = V_0\, T^{20} = [\ 3/7 \quad 4/7\ ]$.

After 21 years, $V_{21} = V_0\, T^{21} = [\ 3/7 \quad 4/7\ ]$; market shares are stable and did not change.

The market share after 20 years has stabilized to $[\ 3/7 \quad 4/7\ ]$. This means that

$$[\ 3/7 \quad 4/7\ ] \begin{bmatrix} .60 & .40 \\ .30 & .70 \end{bmatrix} = [\ 3/7 \quad 4/7\ ]$$

Once the market share reaches an equilibrium state, it stays the same, that is, ET = E.

This helps us answer the next question.